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THE HOMOLOGY OF $MO(1)^{\wedge\infty}$ AND $MU(1)^{\wedge\infty}$

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Let $(\mathbb{Z}/2)^\infty$ denote the subgroup of 0 consisting of the diagonal matrices; let $MO(1)^{\wedge\infty}$ denote the Thom spectrum over $B(\mathbb{Z}/2)^\infty$. Thus $MO(1)^{\wedge\infty}$ represents the bordism theory $h_*^{(\mathbb{Z}/2)^\infty}(-)$ of manifolds with stable normal bundle given a splitting as an ordered sum of line bundles. Similarly let $MU(1)^{\wedge\infty}$ be the Thom spectrum over $B(S^1)^\infty$.

In this note we prove:

Theorem 1. *$H_*(MO(1)^{\wedge\infty};\mathbb{Z}/2)$ is a free (i.e. extended) comodule over the dual of the Steenrod algebra A_*^2; hence $MO(1)^{\wedge\infty}$ splits as a wedge of (suspensions of) Eilenberg–MacLane spectra $H\mathbb{Z}/2$.*

Theorem 2. *$H_*(MU(1)^{\wedge\infty};\mathbb{Z}/p)$ is a free comodule over $H_*BP\subset A_*^p$ for any prime p; hence, since $H_*(MU(1)^{\wedge\infty};\mathbb{Z})$ is torsion free, $MU(1)^{\wedge\infty}$ splits p-locally as a wedge of (suspensions of) Brown–Peterson spectra BP.*

The forgetful functor

$$\Phi : h_*^{(\mathbb{Z}/2)^\infty}(-)\to N_*(-)$$

is a surjection on the point-rings – this is because $N_*(\text{pt.})$ has a set of algebra generators (real projective spaces and hypersurfaces thereof) lying in the image of Φ – so Theorem 1 entails the following:

Corollary. *$\Phi : h_*^{(\mathbb{Z}/2)^\infty}(X)\to N_*(X)$ is surjective for all spaces X.*

The second author would like to take the opportunity to apologise for his erroneous claim to exhibit an X contradicting this in [2]. R.E. Strong pointed out the error and gave an elegant proof of the corollary using the splitting principle. However this left open the question as to whether $H_*(MO(1)^{\wedge\infty};\mathbb{Z}/2)$ was actually free over A_*^2. What makes the question hard is the lack of any geometrically defined product on $MO(1)^{\wedge\infty}$ possessing a unit (it has card{linear isometries: $\mathbb{R}^\infty \oplus \mathbb{R}^\infty \to \mathbb{R}^\infty$} such products which do not!) – this means that the Milnor–Moore theorem, for example, does not apply. We answer the question here using the theorem of Adams and Margolis [1] (or Moore and Peterson [4] in the complex case) together with some bare-handed algebra to show that these theorems apply.

We do not know if the complex analogue of the corollary is true. To prove this it would suffice to find a set of generators for $U_*(\mathrm{pt.})$ having stable normal bundles split into a sum of complex line bundles.

Proof of Theorem 1. Let $M(k)^*$ denote $H^{*+k}(MO(1)^{\wedge k}; \mathbb{Z}/2)$ so that we have an inverse system

$$M(1)^* \xleftarrow{\varepsilon_1} M(2)^* \xleftarrow{\varepsilon_2} M(3)^* \xleftarrow{\varepsilon_3} \cdots \tag{A}$$

in which each ε_i is surjective and the limit of the system is $H^*(MO(1)^{\wedge\infty}; \mathbb{Z}/2) = M(\infty)^*$ say.

The key to Theorem 1 is the theorem of Adams and Margolis [1]. To state this result let $P_t(i)$ denote the element of Milnor's basis for A_2^* dual to ξ_t^i and let $P_t^s = P_t(2^s)$; then if $s<t$, $(P_t^s)^2 = 0$ and their theorem says that a connective (i.e. bounded below) A_2^*-module M, say, is free if and only if

$$H(M, P_t^s) = 0 \quad \text{for all } t \text{ and } s<t.$$

Proposition 3. *Given any integer $r>0$, there is an integer $k_r = k_r(s,t)$ $(s<t)$, such that $H(M(k)^*, P_t^s) = 0$ in degrees $\le r$ for all $k \ge k_r$.*

Given Proposition 3 it is an easy argument to show that $H(M(\infty)^*, P_t^s) = 0$ for all t and $s<t$ (for example, by using the duality we sketch below). Thus, by Adams and Margolis, $M(\infty)^*$ is free, i.e. it has a subgroup B^* such that the composite

$$A_2^* \otimes B^* \xrightarrow{1 \otimes \text{inclusion}} A_2^* \otimes M(\infty)^* \xrightarrow{\text{action}} M(\infty)^* \tag{B}$$

is an isomorphism.

Give $M(\infty)^*$ the inverse limit topology arising from (A) – where all the $M(k)^*$ $(k<\infty)$, are assumed to have the discrete topology, as is the ground field $\mathbb{Z}/2$. Then $H_*(MO(1)^{\wedge\infty}; \mathbb{Z}/2) = M(\infty)_*$, say, is the continuous dual of $M(\infty)^*$. Now since A_2^* is locally finite-dimensional, it follows that

$$A_2^* \otimes M(\infty)^* \simeq \varprojlim (A_2^* \otimes M(k)^*) \quad \text{and} \quad A_2^* \otimes B^* \simeq \varprojlim (A_2^* \otimes B(k)^*)$$

(where $B(k)^* = \mathrm{Im}(B^* \to M(k)^*)$).

Hence the isomorphism (B) dualises to an isomorphism

$$M(\infty)_* \xrightarrow{\text{action}_*} A_*^2 \otimes M(\infty)_* \xrightarrow{1 \otimes \text{projection}} A_*^2 \otimes B_* \tag{C}$$

where B_* is dual to B^*; it is easy to verify that 'action$_*$' is the same as the

A^2_*-coaction on $M(\infty)_* \simeq H_*(MO(1)^{\wedge\infty}; \mathbb{Z}/2)$ so that (C) displays $M(\infty)_*$ as a free comodule. Thus Proposition 3 implies Theorem 1.

Proof of Proposition 3. Let $H^*((\mathbb{R}P^\infty)^k; \mathbb{Z}/2) = \mathbb{Z}/2[x_1, x_2, \ldots, x_k]$ where x_i generates H^1 of the ith factor. We may identify $M(k)^*$ with the ideal in $\mathbb{Z}/2[x_1, x_2, \ldots, x_n]$ generated by the Thom class $u_k = x_1 x_2 \cdots x_k$.

To compute the action of the $P_t(i)$ on $M(k)^*$ we use the following easily verified formulae:

$$P_t(i)x = \begin{cases} x^{2^t} & i=1, \\ 0 & i>1. \end{cases} \tag{1)(i}$$

$$P_t(i)x^{2^t} = 0 \quad 1 \le i < 2^t. \tag{1)(ii}$$

For any cohomology classes a, b

$$P_t(i)(ab) = \sum_j (P_t(j)a)(P_t(i-j)b). \tag{2}$$

The proof proceeds by induction on r for fixed s and t (with $s<t$) and we shall prove the result with

$$\begin{aligned} k_r &= 2^s + \tfrac{1}{2}r(r+1) + r2^s(2^t-1) \\ &= 2^s + \sum_{j=1}^{r} (j + 2^s(2^t-1)) \end{aligned}$$

(which is not necessarily the sharpest possible estimate – certainly not if $s=0$, in which case P_t^s is a derivation and then the Künneth theorem gives precise results).

First case: $r=0$. $M(k)^0$ is generated by u_k and $P_t^s(u_k) \ne 0$ for $k \ge 2^s$ so we are done.

Inductive step. Assume the proposition for $(r-1)$ with

$$k_{r-1} = 2^s + \sum_{j=1}^{r-1} (j + 2^s(2^t-1)).$$

Let $y \in M(k)^r$ with

$$k \ge k_r = 2^s + \sum_{j=1}^{r} (j + 2^s(2^t-1))$$

and assume $P_t^s(y) = 0$. We may regard y as an element of $M(q)^* \otimes M(k-q)^*$ for any $q: 1 \le q \le k$.

First claim. *If* $q \le r + 2^s(2^t-1)$ *(so that* $k - q \ge k_{r-1}$*) then* y *is* P_t^s*-homologous to an element* $\bar{y} \in M(k)^*$ *in which* $x_1, \ldots, x_q$ *occur only with exponent* 1 *or* 2^t.

Proof of claim. To make the method clear we first consider the case $q=1$ and write

$$y = x_1 y_1 + \cdots + x_1^n y_n \quad (y_1, \ldots, y_n \in M(k-1)^*).$$

Let $x_1^m y_m$ be the first non-zero term other than (possibly) $x_1 y_1$ or $x_1^{2^t} y_{2^t}$. Now by properties (1) and (2) above the coefficient of x_1^m in $P_t^s y$ is $P_t^s y_m$, and from $P_t^s y=0$ we deduce that $P_t^s y_m=0$. Since $m>1, y_m$ has degree $\leq r-1$ and so the inductive hypothesis gives us a z_m with $P_t^s z_m=y_m$. Then $y+P_t^s(x_1^m z_m)$ has as its first non-zero term other than $x_1 y_1$ or $x_1^{2^t} y_{2^t}$ some term $x_1^{m'} y_{m'}$ with $m'>m$. Repeating the process gives the result.

In the general case of $q\leq r+2^s(2^t-1)$, let x_p be the first of $x_1, \ldots, x_q$ to occur with exponent other than 1 or 2^t in y, let x_p^m be the lowest such power of x_p occurring, and let the term of least total degree in $x_1, \ldots, x_p$ in which x_p^m occurs be

$$x_1 \cdots x_l x_{l+1}^{2^t} \cdots x_{p-1}^{2^t} x_p^m y' + \text{similar terms} \tag{$*$}$$

(where the 'similar terms' have the same total degree in $x_1, \ldots, x_p$ and all contain x_p with exponent m).

Now $P_t^s y'=0$ since this is the coefficient of $x_1 \cdots x_l x_{l+1}^{2^t} \cdots x_{p-1}^{2^t} x_p^m$ in $P_t^s y$; the inductive hypothesis therefore gives us a z' with $P_t^s z'=y'$ (since y' has degree $\leq r-1$). Then

$$y+P_t^s(x_1 \cdots x_l x_{l+1}^{2^t} \cdots x_{P-1}^{2^t} x_P^m z')$$

has zero as its term corresponding to y'. Since the 'similar terms' in the expression ($*$) are unchanged we can remove each of them by repeating the process, and thus inductively obtain $\bar{y}$ of the form claimed.

Second claim. *If $q\leq r+2^s(2^t-1)$ then y is P_t^s-homologous to an element of the form*

$$s_0 y_0+s_1 y_1+\cdots+s_q y_q \in M(q)^* \otimes M(k-q)^*$$

where each $y_j \in M(k-q)^*$ and s_j denotes the monomial symmetric polynomial containing $x_1^{2^t} x_2^{2^t} \cdots x_j^{2^t} x_{j+1} \cdots x_q$.

Proof of claim. By the *first claim* we may assume y to be of the form

$$y=x_1 \cdots x_q y_0+x_1^{2^t} x_2 \cdots x_q y_{1,1}+\cdots+x_1 x_2 \cdots x_q^{2^t} y_{1,q}+\text{higher terms}$$

(where the 'higher terms' have higher total degree in $x_1, \ldots, x_q$).

Then

$$P_t(2^s-1)y+P_t^s y_{1,1}=0 \quad (\text{coefficient of } x_1^{2^t} x_2 \cdots x_q \text{ in } P_t^s y),$$
$$\ldots$$
$$P_t(2^s-1)y+P_t^s y_{1,q}=0 \quad (\text{coefficient of } x_1 x_2 \cdots x_q^{2^t} \text{ in } P_t^s y),$$

and thus

$$P_t^s(y_{1,1}+y_{1,j})=0 \quad \text{for each } 2\leq j\leq q.$$

By the inductive hypothesis we deduce that there exist $z_{1,j}$ with

$$y_{1,1}+y_{1,j}=P_t^s z_{1,j} \quad \text{for each } 2\leq j\leq q.$$

Then

$$y+P_t^s(x_1x_2^{2^t}\cdots x_qz_{1,2}+\cdots+x_1x_2\cdots x_q^{2^t}z_{1,q})$$

has the form

$$x_1\cdots x_qy_0+(x_1^{2^t}x_2\cdots x_q+\cdots+x_1x_2\cdots x_q^{2^t})y_{1,1}+\text{higher terms}$$

that is,

$$s_0y_0+s_1y_1+\text{higher terms}.$$

Suppose now that we have shown y to be P_t^s-homologous to an element of the form

$$s_0y_0+s_1y_1+\cdots+s_{m-1}y_{m-1}+\text{higher terms}.$$

The 'higher terms' of least total degree in $x_1,\ldots,x_q$ have the form

$$x_1^{2^t}x_2^{2^t}\cdots x_m^{2^t}x_{m+1}\cdots x_qy_{m,1}+\cdots+x_1\cdots x_{q-m}x_{q-m+1}^{2^t}\cdots x_q^{2^t}y_{m,n}$$

(where $n=\binom{q}{m}$).

By equating to zero the coefficient of $x_1^{2^t}\cdots x_m^{2^t}x_{m+1}\cdots x_q$ in P_t^sy we obtain $P_t^sy_{m,1}$ as a function of $s_0, y_0,\ldots,s_{m-1}, y_{m-1}$ (just as we obtained $P_t^sy_{1,1}=P_t(2^s-1)y_0$ in the first case). From the symmetry of this function in $x_1,\ldots,x_q$ we deduce that

$$P_t^sy_{m,1}=\cdots=P_t^sy_{m,n}$$

and so

$$P_t^s(y_{m,1}+y_{m,j})=0 \quad 2\le j\le n.$$

By the inductive hypothesis there exist $z_{m,j}$ with

$$y_{m,1}+y_{m,j}=P_t^sz_{m,j} \quad 0\le j\le n.$$

Then

$$y+P_t^s(x_1^{2^t}\cdots x_{m-1}^{2^y}x_mx_{m+1}^{2^t}x_{m+2}\cdots x_qz_{m,2}+\cdots+x_1\cdots x_{q-m}x_{q-m+1}^{2^t}\cdots x_q^{2^t}z_{n,n})$$

has the form

$$s_0y_0+\cdots+s_my_m+\text{higher terms}$$

completing the inductive proof of the *second claim*.

Finally our inductive step in Proposition 3 follows easily from the second claim for it shows that y is P_t^s-homologous to an element of $N(q)^*\otimes M(k-q)^*$, where

$$N(q)^*=H^{*+q}(\mathrm{MO}(q);\ \mathbb{Z}/2)\subset H^{*+q}(\mathrm{MO}(1)^{\wedge k};\ \mathbb{Z}/2)=M(q)^*;$$

since $N(q)^*$ is free over A_2^* in degrees $\le q$, so is $N(q)^*\otimes M(k-q)^*$ so that $H(N(q)^*\otimes M(k-q)^*, P_t^s)=0$ in degrees $\le q-2^s(2^t-1)$ (by the easy half of the Adams–Margolis theorem) and taking $q=r+2^s(2^t-1)$ we are done.

The proof of Theorem 2 is quite analogous. In place of the theorem of Adams and Margolis we use that of Moore and Peterson [4] which states that with P_t^s

defined as in the case $p=2$ (i.e. P_t^s is dual to $\xi_t^{p^s}$) and if $'A_p^*$ denotes the subalgebra of A_p^* generated by the reduced powers, then an $'A_p^*$-module M is free if and only if $H(M, P_t^s)=0$ (where, if $d: M \to M$ has $d^p=0$, they define $H(M, d)=\ker d/\operatorname{im} d^{p-1}$). Since $'A_p^*=H^*\mathrm{BP}$, Theorem 2 follows from the mod p analogue of Proposition 3 and the duality outlined above, when p is odd. It is an easy exercise to check that the proof of Proposition 3 goes through in the same way for an appropriate k_r, replacing 2 by the odd prime p and $\operatorname{Im}(P_t^s)$ by $\operatorname{Im}(P_t^s)^{p-1}$. The assertion of Theorem 2 for $p=2$ is easy using Theorem 1 and the squaring technique of Liulevicius [3].

Remarks. (a) It is unfortunate that we cannot give precise information about the degree of freeness of $M(k)^*$ over A_2^*. Even if we had the best possible k_r, the results of Adams and Margolis would only tell us that $M(k)^*$ was free up to degree $r-c_r$ where c_r is a constant which is apparently hard to compute.

(b) It would be nice to be able to work in homology throughout; however, dualising [1] and [4] without the too-restrictive hypothesis of locally-finite dimension seems to be non-trivial.

References

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